Fall 2021

EECS 360: Signal and System Analysis

Lab#9 – Sampling, Nyquist Theorem and Aliasing

Simply, Sampling is the first step of the process to convert an analog signal into a digital one. We can represent a continuous time signal as follows:

$$x(t) = A\cos(\omega t)$$

However, digital computers and computer programs cannot process analog signals. Instead they store discrete-time versions of analog signals $x[n] = x(nT_s)$. This is because digital computers can only store discrete numbers. We can obtain a discrete-time signal by sampling a continuous-time signal at equally spaced time instants, $t_n = nT_s$

$$x[n] = x(nT_s) \quad -\infty < n < \infty$$

The individual values $x[n]$ are called the samples of the continuous time signal $x(t)$. The fixed time interval between samples, T_s is also expressed in terms of a sampling rate f_s fs (in samples per second) such that:

$$f_s = \frac{1}{T_s} \quad samples / sec$$

We can define a normalized frequency for the discrete sinusoidal signal.

$$x[n] = x(nT_s) \quad -\infty < n < \infty$$
$$x[n] = A\cos(\omega n T_s) = A\cos(\bar{\omega} n)$$
$$where, \bar{\omega} = \omega T_s = \frac{\omega}{f_s}$$

Here, $\bar{\omega}$ is the normalized or discrete – time frequency. Since we can have different signals with the same $\bar{\omega}$, then there can be an infinite number of continuous-time signal which yield the same discrete-time sinusoid!

Now, two big factors/questions are there when we do sampling:

1. How many samples are enough to represent a continuous time signal?
2. Can a set of samples be represent more than one continuous-time signal?

Claude Elwood Shannon an American mathematician, electrical engineer, and cryptographer known as "the father of information theory" gave an idea how we can sample any continuous signal so that we can reproduce it again based on those samples.

This theorem is called "***Shannon's Sampling Theorem***"***: "A continuous-time signal*** $x(t)$ ***with frequencies no higher than*** $f_{\max}$ ***can be reconstructed exactly from its samples*** $x[n] = x(nT_s)$ ***, if the samples are taken a rate*** $f_s = \frac{1}{T_s}$ ***that is greater than*** $2 \times f_{\max}$***"***

It means we have to choose the "Sampling Frequency, f_s" in such a way that, $f_s > 2 \times f_{\max}$ where $f_{\max}$ is called original signal frequency. This $2 \times f_{\max}$ is the minimum sampling rate and knows as "**Nyquist**" rate.

(i) If we take more samples than "**Nyquist**" rate, we call it oversampling
(ii) If we take lesser samples than "**Nyquist**" rate, we call it oversampling. In this condition we can't reproduce our original signal.

So, Shannon's theorem tell us that if we have at least 2 samples per period of a sinusoid, we have enough information to reconstruct the sinusoid of frequency 1.

So, Nyquist theorem is the answer of our 1st factor/question. Now, what about the 2nd factor/question: Can a set of samples be represent more than one continuous-time signal?

Suppose we have two signals:
$$x_1(t) = A\cos(\omega t) = A\cos(2\pi f t) = A\cos(2\pi \times 0.2 \times t)$$
$$x_2(t) = A\cos(\omega t) = A\cos(2\pi f t) = A\cos(2\pi \times 1.2 \times t)$$

If we samples these signals using 1 sample/sec then we can write,

$$x_1[n] = A\cos(2\pi \times 0.2 \times nT_s) = A\cos(2\pi \times 0.2 \times n \times 1) = A\cos(0.4\pi n)$$
$$x_2[n] = A\cos(2\pi \times 1.2 \times n \times 1) = A\cos(2.4\pi n) = A\cos(0.4\pi n + 2\pi n) = A\cos(0.4\pi n)$$

This example illustrates that, two sampled sinusoids can produce the same discrete-time signal. When this occurs we say that that these signals are aliases of each other. If anything like this happens in our signal reconstruction, we call it "Aliasing" problem.

Lab Assignment

In this lab, we have to perform "Sampling" and then have to find out the effect of Aliasing.

1. Write a MATLAB code for a cos signal and sample the signal in under sampling, equal to Nyquist rate sampling and over sampling rate. Choose a sampling and frequency of your own. Plot all 4 signals in a same graph using subplot command.
2. Find the Fourier transforms of the sinusoidal signal with various sampling rates, and graph them individually. (Note: define frequency vectors properly for each signal.

3. Consider an analog signal $X_a(t) = \cos(20\pi t), 0 \le t \le 1$. Sample this signal at $T_S = 0.01, 0.05, \& 0.1$ second intervals.

(a) What is the frequency of $X_a(t)$?

(b) Produce a stem plot of all three sampled sequences. Use 3 subplots contained in a single figure.

(c) From the stem plots, comment on which sequences are over sampled, under sampled, or ideally sampled.

Questions

1. Explain "Nyquist Theorem" and Aliasing.
2. What happens if we sample at a rate which is less than the Nyquist Rate?
3. Can you explain "Nyquist Theorem" in terms of signal bandwidth?
4. There are three basic steps to convert an analog signal into a digital one. Sampling is the first step. Can you mention anything about the other two steps? (This question is not lab related, but I am more than happy to give you extra credit if you try to answer the question)